Prescription for experimental determination of the dynamics of a quantum black box

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Abstract. We give an explicit way to experimentally determine the evolution operators which completely describe the dynamics of a quantum-mechanical black box: an arbitrary open quantum system. We show necessary and sufficient conditions for this to be possible and illustrate the general theory by considering specifically, one- and two-quantum-bit systems. These procedures may be useful in the comparative evaluation of experimental quantum measurement, communication and computation systems.

1. Introduction

Consider a black box with an input and an output. Given that the transfer function is linear, if the dynamics of the box are described by classical physics, well known situations exist to determine completely the response function of the system. Now consider a quantum-mechanical black box whose input may be an arbitrary quantum state (in a finite-dimensional Hilbert space), with internal dynamics and an output state (of same dimension as the input) determined by quantum physics. The box may even be connected to an external reservoir or have other inputs and outputs which we wish to ignore. Can we determine the quantum transfer function of the system?

The answer is yes. Simply stated, the most arbitrary transfer function of a quantum black box is to map one density matrix into another, \( \rho_{in} \rightarrow \rho_{out} \), and this is determined by a linear mapping \( \mathcal{E} \) which we shall give a method for obtaining. The interesting observation is that this black box may be an attempt to realize a useful quantum device. For example, it may be a quantum cryptography channel [1, 2] (which might include an eavesdropper!), a quantum computer in which decoherence occurs, limiting its performance [3, 4], or just an imperfect quantum logic gate [5, 6], whose performance you wish to characterize to determine its usefulness.

How many parameters are necessary to describe a quantum black box acting on an input with a state space of \( N \) dimensions? How may these parameters be experimentally determined? Furthermore, how is the resulting description of \( \mathcal{E} \)
useful as a performance characterization? These questions have been considered earlier by numerous researchers, including D' Ariano et al. [7], and in greater detail by Jones [8], Turchette et al. [5] and Mabuchi [9].

We give solutions to these questions in this paper. After summarizing the relevant mathematical formalism, we prove that $\mathcal{E}$ may be determined completely by a matrix of complex numbers $\chi$ and provide an accessible experimental method for obtaining $\chi$. We then give explicit constructions for the cases of one and two quantum bits (qubits) and then conclude by describing related performance estimation quantities derivable from $\chi$.

2. State change theory

A general way to describe the state change experienced by a quantum system is by using quantum operations, sometimes also known as superscattering operators or completely positive maps. This formalism has been described in detail in [10] and has been given a brief but informative review in the appendix to [11]. A quantum operation is a linear map $\mathcal{E}$ which completely describes the dynamics of a quantum system:

$$\rho \rightarrow \frac{\mathcal{E}(\rho)}{\text{Tr}[\mathcal{E}(\rho)]},$$

A particularly useful description of quantum operations for theoretical applications is the so-called operator-sum representation [10–13]:

$$\mathcal{E}(\rho) = \sum_j A_j \rho A_j^\dagger.$$

(2)

The $A_j$ are operators acting on the system alone, yet they completely describe the state changes of the system, including any possible unitary operation (quantum logic gate), projection (generalized measurement) or environmental effect (decoherence). In the case of a 'non-selective' quantum evolution, such as arises from uncontrolled interactions with an environment (as in the decoherence of quantum computers), the $A_j$ operators satisfy an additional completeness relation

$$\sum_j A_j^\dagger A_j = I.$$

(3)

This relation ensures that the trace factor $\text{Tr}[\mathcal{E}(\rho)]$ is always equal to one, and thus the state change experienced by the system can be written as

$$\rho \rightarrow \mathcal{E}(\rho).$$

(4)

Such quantum operations are in a one-to-one correspondence with the set of transformations arising from the joint unitary evolution of the quantum system and an initially uncorrelated environment [10]. In other words, the quantum operations formalism also describes the master equation and quantum Langevin pictures widely used in quantum optics [14, 15], where the system's state change arises from an interaction Hamiltonian between the system and its environment [9].

The major limitation of this picture is that the formalism assumes that the input state $\rho$ is in a product state with the rest of the world. Otherwise, if the initial system were to be in a correlated state with part of the quantum black box (for example, if it were entangled with an environment shared with the black box), then equation (2) would not be valid, and the quantum operations description as defined
above would not be applicable. For the purposes of this paper, we shall define a quantum black box operation as one satisfying equation (2).

Within the context of these physically reasonable assumptions, our goal will be to describe the state change process by determining the operators $A_i$ which describe $E$ (and until section 5 we shall limit ourselves to those which satisfy equation (3)). Once these operators have been determined, many other quantities of great interest, such as the fidelity, entanglement fidelity and quantum channel capacity [16–18] can be determined. Typically, the $A_i$ operators are derived from a theoretical model of the system and its environment; for example, they are closely related to the Lindblad operators. However, what we propose here is different: to determine systematically from experiment what the $A_i$ operators are for a specific quantum black box.

3. General experimental procedure

The experimental procedure may be outlined as follows. Suppose that the state space of the system has $N$ dimensions; for example, $N = 2$ for a single qubit. $N^2$ pure quantum states $|\psi_1\rangle\langle\psi_1|, \ldots, |\psi_{N^2}\rangle\langle\psi_{N^2}|$ are experimentally prepared, and the output state $E(|\psi\rangle\langle\psi|)$ is measured for each input. This may be done, for example, by using quantum state tomography [19–21]. In principle, the quantum operation $E$ can now be determined by a linear extension of $E$ to all states. We prove this below.

The goal is to determine the unknown operators $A_i$ in equation (2). However, experimental results involve numbers (not operators, which are a theoretical concept). To relate the $A_i$ to measurable parameters, it is convenient to consider an equivalent description of $E$ using a fixed set of operators $\hat{A}_n$, which form a basis for the set of operators on the state space, so that

$$A_i = \sum_m a_{im} \hat{A}_m$$  \hspace{1cm} (5)

for some set of complex numbers $a_{im}$. Equation (2) may thus be rewritten as

$$E(\rho) = \sum_{mn} \hat{A}_m \rho \hat{A}_n \chi_{mn},$$ \hspace{1cm} (6)

where $\chi_{mn} = \sum_i a_{im} a_{in}^*$ is a 'classical' error correlation matrix which is positive Hermitian by definition. This shows that $E$ can be completely described by a complex number matrix $\chi$, once the set of operators $\hat{A}_i$ has been fixed. In general, $\chi$ will contain $N^2 - N^2$ independent real parameters, because a general linear map of $N$ by $N$ complex matrices to $N$ by $N$ matrices is described by $N^4$ independent parameters, but there are $N^2$ additional constraints because $\rho$ remains Hermitian with trace one. Algebraically, this fact is expressed by the completeness relation

$$\sum_i \hat{A}_i^\dagger \hat{A}_i + I.$$ \hspace{1cm} (7)

which gives $N^2$ real constraints. Note that the restriction that the map be a quantum operation does not change the counting, since by Choi’s [13] results the set of quantum operations is just the positive cone in the real vector space of Hermitian-preserving maps, and the positive cone of a real vector space has the
same dimensionality as the underlying vector space. We shall show how to
determine \( \chi \) experimentally and then show how an operator sum representation
of the form (2) can be recovered once the \( \chi \) matrix is known.

Let \( \rho_j, 1 \leq j \leq N^2 \), be a set of linearly independent basis elements for the
space of \( N \times N \) matrices. A convenient choice is the set of operators \( |n\rangle \langle m| \).

Experimentally, the output state \( \mathcal{E}(|n\rangle \langle m|) \) may be obtained by preparing the input
states \( |n\rangle, |m\rangle, |n\rangle = (|n\rangle + |m\rangle)/2^{1/2} \), and \( |n\rangle = (|n\rangle + i|m\rangle)/2^{1/2} \) and forming
linear combinations of \( \mathcal{E}(|n\rangle \langle n|) \), \( \mathcal{E}(|m\rangle \langle m|) \), \( \mathcal{E}(|n\rangle \langle m|) \) and \( \mathcal{E}(|m\rangle \langle n|) \). Thus it is
possible to determine \( \mathcal{E}(\rho_j) \) by state tomography, for each \( \rho_j \).

Furthermore, each \( \mathcal{E}(\rho_j) \) may be expressed as a linear combination of the basis
states:

\[
\mathcal{E}(\rho_j) = \sum_k \lambda_{jk} \rho_k. \tag{8}
\]

Since \( \mathcal{E}(\rho_j) \) is known, \( \lambda_{jk} \) can thus be determined. To proceed, we may write

\[
A_m \rho_j A_m^\dagger = \sum_k \beta_{jk} \rho_k, \tag{9}
\]

where \( \beta_{jk} \) are complex numbers which can be determined by standard algorithms
given the \( A_m \) operators and the \( \rho_j \) operators. Combining the last two expressions we have

\[
\sum_k \sum_n \lambda_{mn} \beta_{jk} \rho_k = \sum_k \lambda_{jk} \rho_k. \tag{10}
\]

From independence of the \( \rho_k \) it follows that, for each \( k \),

\[
\sum_n \beta_{jk} \lambda_{mn} = \lambda_{jk}. \tag{11}
\]

This relation is a necessary and sufficient condition for the matrix \( \chi \) to give the
correct quantum operation \( \mathcal{E} \). One may think of \( \chi \) and \( \lambda \) as vectors, and \( \beta \) as a
\( N^4 \times N^4 \) matrix with columns indexed by \( mn \), and rows by \( ij \). To show how \( \chi \) may
be obtained, let \( \kappa \) be the generalized inverse for the matrix \( \beta \), satisfying the relation

\[
\beta_{jk} = \sum_{\alpha, \lambda} \beta_{jk} \kappa_{\alpha, \lambda} \beta_{\alpha, \lambda}. \tag{12}
\]

Most computer algebra packages are capable of finding such generalized inverses.
In appendix A it is shown that \( \chi \) defined by

\[
\chi_{mn} = \sum_{jk} \kappa_{jk} \lambda_{jk}, \tag{13}
\]
satisfies equation (11). The proof is somewhat subtle, but it is not relevant to the
application of the present algorithm.

Having determined \( \chi \), one immediately obtains the operator sum representation
for \( \mathcal{E} \) in the following manner. Let the unitary matrix \( U^\dagger \) diagonalize \( \chi \):

\[
\chi_{mn} = \sum_{\alpha, \lambda} U_{mn} \delta_{\alpha, \lambda} U_{\alpha, \lambda}^\dagger. \tag{14}
\]

From this it can easily be verified that

\[
A_i - d_i^{1/2} \sum_j U_{ij} \hat{A}_j. \tag{15}
\]
gives an operator-sum representation for the quantum operation $\mathcal{E}$. Our algorithm
may thus be summarized as follows: $\lambda$ is experimentally measured and, given $\beta$, determined by a choice of $\mathcal{A}$, we find the desired parameters $\chi$ which completely
describe $\mathcal{E}$.

4. One and two quantum bits

The above general method can be simplified in the case of one- and two-qubits
operations to provide explicit formulae which may be useful in the experimental context. This simplification is made possible by choosing the fixed operators $\mathcal{A}_i$ to have commutation properties which conveniently allow the $\chi$ matrix to be determined by straightforward matrix multiplication. In the one-qubit case, we use

$$\mathcal{A}_0 = I,$$

$$\mathcal{A}_1 = \sigma_x,$$

$$\mathcal{A}_2 = -i\sigma_y,$$

$$\mathcal{A}_3 = \sigma_z,$$

where the $\sigma_i$ are the Pauli matrices. There are 12 parameters, specified by $\chi$, which
determine an arbitrary single-qubit black-box operation $\mathcal{E}$; three of these describe arbitrary unitary transforms $\exp(i\sum_r r_\lambda \sigma_\lambda)$ on the qubit, and nine parameters describe possible correlations established with the environment $E$ via $\exp(i\sum_r \gamma_\lambda \sigma_\lambda \otimes \sigma_\lambda^E)$. Two combinations of the nine parameters describe physical processes analogous to the $T_1$ and $T_2$ spin–lattice and spin–spin relaxation rates familiar to use from classical magnetic spin systems. However, the dephasing and energy loss rates determined by $\chi$ do not completely characterize the decoherence; the decoherence of a single qubit must be described by more than just two
parameters. Nine are needed, in addition to the three necessary to describe the unitary evolution, for a total of 12.

These 12 parameters may be measured using four sets of experiments. As a
specific example, suppose the input states $|0\rangle, |1\rangle, +\rangle = (|0\rangle + |1\rangle)/2$ and
$-\rangle = (|0\rangle - |1\rangle)/2$ are prepared, and the four matrices

$$\rho'_1 = \mathcal{E}(|0\rangle\langle 0|),$$

$$\rho'_2 = \mathcal{E}(|1\rangle\langle 1|),$$

$$\rho'_3 = \mathcal{E}(+\rangle\langle -|) - i\mathcal{E}(-\rangle\langle +|) = \frac{1 - i}{2}(\rho'_1 - \rho'_4),$$

$$\rho'_4 = \mathcal{E}(+\rangle\langle +|) - i\mathcal{E}(-\rangle\langle -|) = \frac{1 + i}{2}(\rho'_1 + \rho'_4),$$

are determined using state tomography. These correspond to $\rho'_j = \mathcal{E}(\rho_j)$, where

$$\rho_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\rho_2 = \rho_1 \sigma_x, \rho_3 = \sigma_x \rho_1, \text{ and } \rho_4 = \sigma_x \rho_3 \sigma_x.$$ From equation (9) and equations (16)–(19)
we may determine $\beta$, and similarly $\rho'_j$ determines $\lambda$. However, because of the
particular choice of basis, and the Pauli matrix representation of $A_i$, we may express the $\beta$ matrix as the Kronecker product $\beta = A \otimes I$, where

$$A = \frac{1}{2} \begin{bmatrix} i & \sigma_z \\ \sigma_x & -I \end{bmatrix},$$

so that $\chi$ may be expressed conveniently as

$$\chi = A \begin{bmatrix} \rho'_1 \\ \rho'_2 \\ \rho'_3 \\ \rho'_4 \end{bmatrix} A^T,$$

in terms of block matrices.

Likewise, it turns out that the parameters $\chi_2$ describing the black-box operations on two qubits can be expressed as

$$\chi_2 = A_2 \tilde{\rho}' A_2,$$

where $A_2 = A \otimes I$, and $\tilde{\rho}'$ is a matrix of 16 measured density matrices:

$$\tilde{\rho}' = P^{T} \begin{bmatrix} \rho'_{11} & \rho'_{12} & \rho'_{13} & \rho'_{14} \\ \rho'_{21} & \rho'_{22} & \rho'_{23} & \rho'_{24} \\ \rho'_{31} & \rho'_{32} & \rho'_{33} & \rho'_{34} \\ \rho'_{41} & \rho'_{42} & \rho'_{43} & \rho'_{44} \end{bmatrix} P,$$

where $\rho'_{nm} = \mathcal{E}(\rho_{nm}), \rho_{nm} = T_q 00)(00)T_m$, $T_1 = I \otimes I$, $T_2 = I \otimes \sigma_z$, $T_3 = \sigma_z \otimes I$, $T_4 = \sigma_z \otimes \sigma_z$, and $P = I \otimes (\rho_{00} + \rho_{12} - \rho_{21} + \rho_{33}) \otimes I$ is a permutation matrix. Similar results hold for $k$ greater than two qubits. Note that, in general, a quantum black box acting on $k$ qubits is described by $16^k - 4^k$ independent parameters.

There is a particularly elegant geometric view of quantum operations for a single qubit. This is based on the (real three-dimensional) Bloch vector $\mathbf{i}$ which is defined by

$$\rho = I - \mathbf{i} \cdot \sigma,$$

satisfying $|\mathbf{i}| \leq 1$, where $\sigma$ is a vector formed from the three Pauli matrices. The map (4) is equivalent to a map of the form

$$\mathbf{i} \rightarrow \mathbf{i}' = M \mathbf{i} - \mathbf{c},$$

where $M$ is a $3 \times 3$ matrix, and $\mathbf{c}$ is a constant vector. This is an affine map, mapping the Bloch sphere into itself. If the $A_i$ operators are written in the form

$$A_i = \alpha_i I + \sum_{k=1}^3 a_{ik} \sigma_k,$$

then it is not difficult to check that

$$M_{jk} = \sum_{i} \left[ a_{ij} a_{ik}^* + a_{ik}^* a_{ij} + \left( |\alpha_i|^2 - \sum_{p} a_{ip} a_{ip}^* \right) \delta_{jk} + i \sum_p \epsilon_{jlp} (a_{ip} a_{lp}^* - a_{ip}^* a_{lp}) \right],$$

$$c_j = 2 i \sum_{l} \sum_{p} \epsilon_{jlp} a_{ij} a_{lp}^*,$$

where we have made use of equation (3) to simplify the expression for $\mathbf{c}$. 
The meaning of the affine map (30) is made clearer by considering the polar decomposition [22] of the matrix $M$. Any real matrix $M$ can always be written in the form

$$ M = OS, $$

where $O$ is a real orthogonal matrix with determinant 1, representing a proper rotation, and $S$ is a real symmetric matrix. Viewed this way, the map (30) is just a deformation of the Bloch sphere along principal axes determined by $S$, followed by a proper rotation due to $O$, followed by a displacement due to $c$. Various well known decoherence measures can be identified from $M$ and $c$; for example, $T_1$ and $T_2$ are related to the magnitude of $c$ and the norm of $M$. Other measures are described in the following section.

5. Applications

Systematic methods for experimentally determining a complete description of an unknown quantum operation are described. For a $k$ qubit operation, this gives $16^k - 4^k$ parameters, a huge amount of data! What is the usefulness of all this information? We demonstrate the utility of knowing $\chi$ below, using two examples which return to the one- and two-qubit cases. We then describe how $\chi$ is related to important quantities in quantum information theory, which allow evaluation of the fidelity of an experimental implementation.

Consider a one-qubit black box of unknown dynamics $E_1$. Suppose that the following four density matrices are obtained from experimental measurements, performed according equations (20)–(23):

$$ \rho'_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, $$

$$ \rho'_2 = \begin{bmatrix} 0 & (1 - \gamma)^{1/2} \\ 0 & 0 \end{bmatrix}, $$

$$ \rho'_3 = \begin{bmatrix} 0 & 0 \\ (1 - \gamma)^{1/2} & 0 \end{bmatrix}, $$

$$ \rho'_4 = \begin{bmatrix} \gamma & 0 \\ 0 & 1 - \gamma \end{bmatrix}. $$

where $\gamma$ is a numerical parameter. From an independent study of each of these input–output relations, one could make several important observations: the ground state $|0\rangle$ is left invariant by $E_1$, the excited state $|1\rangle$ partially decays to the ground state, and superposition states are damped. This would suggest that both $T_1$ and $T_2$ processes are at work in this black box.

However, let us proceed systematically and determine $\chi$ using this data. From equations (25) and (26), we find the $\chi$ matrix for this process to be

$$ \chi = \begin{bmatrix} 1 + (1 - \gamma)^{1/2}^2 & 0 & 0 & \gamma \\ 0 & \gamma & -\gamma & 0 \\ 0 & -\gamma & \gamma & 0 \\ \gamma & 0 & 0 & [1 - (1 - \gamma)^{1/2}^2] \end{bmatrix}. $$


Using equations (14) and (15), we then obtain (after a little simplification) the $A_{i}$ which give the operator sum representation for this quantum operation:

$$ A_0 = \begin{bmatrix} 1 & 0 \\ 0 & (1 - \gamma)^{1/2} \end{bmatrix}, \quad (40) $$

$$ A_1 = \begin{bmatrix} 0 & \gamma^{1/2} \\ \gamma^{-1/2} & 0 \end{bmatrix}. \quad (41) $$

These operators define a well known process called amplitude damping. It can result from a relaxation process with a microscopic interaction Hamiltonian of the form $\mathcal{H}_1 = \gamma'(\sigma^{-}\sigma^{-} + \sigma^{+}\sigma^{+})$, where $\sigma^{-}$ and $\sigma^{+}$ are system and environment creation operators, and $\gamma'$ is related to $\gamma'$ and the interaction time. This description of $\mathcal{E}_1$ captures the fact that, for a qubit, relaxation is a process which cannot be described as a combination of independent $T_1$ or $T_2$ process. This is important, for instance, in quantum error correction, where one wishes to reverse the effects of such decoherence, because better codes exist to correct amplitude damping than for general $T_1$ and $T_2$ processes [23, 24].

The dynamics of a two-qubit quantum black box $\mathcal{E}_2$ pose an even greater challenge for our understanding. Of the 240 parameters which describe the $\chi$ matrix for $\mathcal{E}_2$, 15 determine the unitary operation performed to the two qubits, 90 describe single-qubit interactions with the environment, and 135 describe two-qubit interactions with the environment. Certainly each of these groups of numbers gives us some useful information, but perhaps what is most useful of all are fidelity measures, as illustrated in the following example.

An important primitive element in quantum computation is the controlled-NOT gate [25]. An implementation of it using trapped ions has been reported [6], but how do we know that this experimental manifestation is truly a quantum logic gate? A systematic procedure is provided by equations (27) and (28); prepare 16 input states $\rho_{in}$, measure the outputs $\rho_{out}$ and calculate the $\chi$ matrix from the result. This matrix fully characterizes any noise process happening in the gate and not only provides an interpretation of the noise process but also measures which can be evaluated to determine whether the implementation is quantum or classical.

The results that can be expected from such a procedure may be studied by considering a noisy controlled-NOT gate [26] modelled by the Hamiltonian

$$ H_{\text{CN}} = H_{\text{CN}}^{0} + \frac{\gamma}{2} (1 - \sigma_{z}^{a}) \sigma_{z}^{a} + \frac{\gamma}{2} (1 - \sigma_{z}^{b}) \sigma_{z}^{b}. \quad (42) $$

where $\sigma^{a}$ and $\sigma^{b}$ are control and target qubit operators, and $\sigma^{e}$ and $\sigma^{h}$ are environment modes for the two qubits. The ideal gate has the Hamiltonian

$$ H_{\text{CN}}^{0} = \frac{\pi}{4} (1 - \sigma_{z}^{a}) (\sigma_{z}^{b} - 1). \quad (43) $$

and the noisy gate transform is $U_{\text{CN}} = \exp(iH_{\text{CN}})$, as has previously been examined in [26]. Note that time is taken to be in units of single gate times, so that it does not appear explicitly in the exponentiation. The phenomenological Hamiltonian $H_{\text{CN}}$ describes a controlled-NOT gate with simultaneous phase damping (of strength $\gamma$) occurring independently for each qubit. Note that each qubit is coupled to just a single-mode environment, but this is sufficient to capture the dynamical behaviour for small $\gamma$. Simulation of $U_{\text{CN}}$ and the measurement of
\( \rho'_{nm} \) give a \( \chi \) matrix which reduces to the following operator sum representation for the process:

\[
A_0 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix},
\]

(44)

\[
A_1 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & -\gamma & 0 & 0 \\
0 & 0 & -i\gamma/\pi & -\gamma/2 \\
0 & 0 & -\gamma/2 & i\gamma/\pi
\end{bmatrix},
\]

(45)

\[
A_2 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\gamma \\
0 & 0 & -\gamma & 0
\end{bmatrix},
\]

(46)

to first order in \( \gamma \) (given non-normalized; higher-order terms are needed to satisfy equation (3)). One interpretation of this result is that, although the gate often works perfectly (\( A_0 \) is the ideal transform), not only sometimes will the inversion of the target qubit be imperfect (when a 'quantum jump' corresponding to \( A_1 \) occurs [26]), but also sometimes the control qubit will collapse, leaving an inverted target qubit (corresponding to \( A_2 \)).

More valuable, perhaps, is the fact that, from this complete description of the quantum process, many other interesting quantities can be evaluated. A quantity of particular importance is the entanglement fidelity [11, 27], which in general can be used to measure how closely the dynamics of the quantum system under consideration approximates that of some ideal quantum system.

Suppose that the target quantum operation is a unitary quantum operation \( U(\rho) = U\rho U^\dagger \), and the actual quantum operation implemented experimentally is \( \mathcal{E} \). The entanglement fidelity can be defined as [27]

\[
F_e(\rho, U, \mathcal{E}) \equiv \sum_i |\text{Tr}(U^\dagger A_i \rho)|^2
\]

\[
= \sum_{mn} \chi_{mn} \text{Tr}(U^\dagger A_m \rho) \text{Tr}(\rho A_n U).
\]

(47)

(48)

The second expression follows from the first using equation (5) and shows that errors in the experimental determination of \( \mathcal{E} \) (resulting from errors in preparation and measurement) propagate linearly to errors in the estimation of entanglement fidelity. The minimum value of \( F_e \) over all possible states \( \rho \) is a single parameter which describes how well the experimental system implements the desired quantum logic gate. In the case of the noisy controlled-NOT gate described above, it can be shown that the minimum entanglement fidelity is

\[
F_e \approx 1 - 1.8\gamma^2.
\]

(49)
One may also be interested in the minimum fidelity of the gate operation. This is given by the expression

\[ F \equiv \min_{\psi} \langle \psi | U^{\dagger} E(\psi) \langle \psi | U | \psi \rangle. \quad (50) \]

where the minimum is over all pure states, \[ | \psi \rangle \]. As for the entanglement fidelity, we may show that this quantity can be determined robustly, because of its linear dependence on the experimental errors. Once again, a numerical search shows that the minimum gate fidelity for the noisy controlled-NOT gate is given by

\[ F \approx 1 - 1.8 \gamma^2. \quad (51) \]

Although this expression is identical with that given above for the minimum entanglement fidelity, the two expressions are not in general equal [16] although they are related [28] as one would expect of good measures of reliability for gate operation.

Our procedure can in principle be used to determine the form of the Lindblad operator \( \mathcal{L} \) used in Markovian master equations of the form

\[ \dot{\rho} = \mathcal{L}(\rho). \quad (52) \]

where for convenience time \( \tau \) is measured in dimensionless units, to make \( \mathcal{L} \) dimensionless. This result follows from the fact that Lindblad operators \( \mathcal{L} \) are just the logarithms of quantum operations; that is, \( \exp \mathcal{L} \) is a quantum operation for any Lindblad operator \( \mathcal{L} \) and \( \log \mathcal{E} \) is a Lindblad operator for any quantum operation \( \mathcal{E} \). In the context of the amplitude damping and controlled-NOT examples discussed above, one could use this observation to determine the appropriate form for \( \mathcal{L} \), assuming that the examples are described by such a Markovian evolution. This calculation is straightforward, but the details are beyond the scope of this paper.

Quantum operations can also be used to describe measurements. For each measurement outcome \( i \), there is associated a quantum operation \( \mathcal{E}_i \). The corresponding state change is given by

\[ \rho \rightarrow \frac{\mathcal{E}_i(\rho)}{\text{Tr} [\mathcal{E}_i(\rho)]}. \quad (53) \]

where the probability of the measurement outcome occurring is \( p_i = \text{Tr} [\mathcal{E}_i(\rho)] \).

Note that this mapping may be nonlinear, because of this renormalization factor.

Despite the possible nonlinearity, the procedure that we have described may be adapted to evaluate the quantum operations describing a measurement. To determine \( \mathcal{E}_i \) we proceed exactly as before, except now we must perform the measurement a sufficiently large number of times that the probability \( p_i \) can be reliably estimated, for example by using the frequency of occurrence of outcome \( i \).

Next, \( \rho_i \) is determined using tomography, allowing us to obtain

\[ \mathcal{E}_i(\rho_i) = \text{Tr} [\mathcal{E}_i(\rho_i)] \rho_i. \quad (54) \]

for each input \( \rho_i \) which we prepare, since each term on the right-hand side is known. Now we proceed exactly as before to evaluate the quantum operation \( \mathcal{E}_i \).

This procedure may be useful, for example, in evaluating the effectiveness of a quantum non-demolition (QND) measurement [29]. This can be illustrated using the controlled-NOT example. Suppose that a black box is constructed to perform
a simple-quantum computation: a Hadamard transform [25] (on the control qubit) followed by a controlled-NOT. This box (which has been certified to be high fidelity using previous tomography experiments) could be used to convert a ground-state input $|00\rangle$ to the EPR state $(|00\rangle + |11\rangle)/\sqrt{2}$. Suppose that appended to this box is another box, which performs a QND measurement on one of the qubits. Its two outputs would be the post-measurement state of the two qubits $\rho_{\text{out}}$ and a ‘red light’ (a classical indicator) that indicates the zero or one result from the measurement. For a perfect QND measurement, both qubits of $\rho_{\text{out}}$ would be in a number eigenstate, by virtue of the entanglement in the EPR state. The question is: how good is the QND measurement box? This question can be answered by performing additional tomography experiments as we have described above. The result would be two $\chi$ matrices, which give the quantum operation performed when the red light is on and when it is off. The advantage of this systematic approach is that it gives not only traditional statistical measures such as probability of proper operation but also a complete analysis of what quantum operation the imperfect QND box actually performs.

6. Conclusion

In conclusion, we have shown how the dynamics of a quantum system may be experimentally determined using a systematic procedure. This elementary system identification step [30] which one might perhaps term quantum process tomography (as a natural extension to quantum state tomography, which as we have mentioned has already been realized experimentally) opens the way for robust experimental determination of a wide variety of interesting quantities. Amongst those that may be of particular interest are the quantum channel capacity, the fidelity and the entanglement fidelity. We expect these results to be of great use in the experimental study of quantum computation, quantum error correction, quantum cryptography, quantum coding and quantum teleportation.

After this paper was submitted, our attention was drawn to a similar paper by Poyatos et al. [31].

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Appendix A: Proof of the $\chi$ relation

The difficulty in verifying that $\chi$ defined by equation (13) satisfies equation (11) is that in general $\chi$ is not uniquely determined by the last set of equations. For
convenience we shall rewrite these equations in matrix form as

\[ \beta \chi = \lambda, \quad (A1) \]
\[ \chi = \kappa \lambda. \quad (A2) \]

From the construction that led to equation (6) we know that there exists at least
one solution to equation (A1), which we shall call \( \chi' \). Thus \( \lambda = \beta \chi' \). The generalized inverse satisfies \( \beta \lambda = \beta \). Pre-multiplying the definition of \( \chi \) by \( \beta \) gives

\[ \beta \chi = \beta \kappa \lambda. \quad (A3) \]
\[ = \beta \kappa \beta \chi'. \quad (A4) \]
\[ = \beta \chi'. \quad (A5) \]
\[ - \lambda. \quad (A6) \]

Thus \( \chi \) defined by equation (A2) satisfies equation (A1), as was required to show.

The necessity to use general inverses arises from the possibility that the set of
operators \( \hat{A} \), spanning the space of operators need not be linearly independent.
This possibility, which may appear somewhat pathological, is in fact sometimes
useful in quantum information theory. Typically, the use of general inverses is not
necessary.

References

Contemp. Phys., 36, 149.
4298.
0702010.